THE PARADOXES OF MATERIAL IMPLICATION

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Abstract

'Paradoxes of material implication' is a significant topic in modern symbolic and mathematical logic. Various attempts have been taken to resolve these paradoxes. Thus, a number of schools of logic have been developed in this regard. In our present paper we examine three of the main schools of modern logic which deal with these paradoxes: many-valued logic, modal logic and relevance logic. Three-valued logic, which is a kind of many-valued logic, fails to show any promise in resolving these paradoxes as it adopts the entire truth table based on traditional bivalence. Five-valued logic, another kind of many-valued logic, shows some promises in resolving these paradoxes. But it destroys the system of propositional calculus and the process of judging the validity/invalidity of arguments. So it is difficult to accept this solution with such a price. Modal logic, the second approach discussed in this paper, resolves these paradoxes by introducing the device of strict implication. But the problem is that it creates some new paradoxes, namely paradoxes of strict implication, which are analogous to the paradoxes of material implication. Hence modal logic is also not adequate. Relevance logic, however, resolves these paradoxes without creating any new paradox. It does not destroy the system of propositional calculus or the process of judging the validity/invalidity of arguments. Hence, relevance logic solves the problems of the logic of implication. Although there are some minor difficulties in relevance logic, we hope that more work in this area will resolve these problems soon, so that relevance logic will provide a fully adequate logic of implication.

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Chapter 1

Introduction: Paradoxes of Material Implication

In symbolic logic, there are four types of compound statements of which conditional statement is more attention grabbing for its comparatively complex structure and truth value. A compound statement that has at least two components that have a relation among them such that one component apparently deduces the other is a conditional statement. That is, in conditional statement two components are conditional to each other. Thus the compound statement 'If it rains, then there will be a flood' is a conditional statement, since it carries two components that have conditional relation. The component that stands prior to 'then' is the antecedent (the implying statement), and the component that stands behind 'then' is the consequent (the implied statement). We should note that the antecedent and consequent, as components of compound statement, are statements in their own right. In symbolic logic, conditional statements are also called 'hypothetical statements', 'implicative statements' or simply 'implications'. My observation is that most logicians simply use the term 'implications'. And that's why, in this paper, I will use the term 'implication' for conditional statement.

Some problems regarding the implication were detected by ancient and medieval logicians.[1] These problems are categorized as paradoxes of material implication

(PMI). In the first three sections of this introduction, we will discuss various types of implications, material implication and the nature of the PMI, respectively. It should be noted here that although PMI were detected by ancient and medieval logicians, no significant development occurred in these periods regarding the issue. In the beginning of twentieth century logicians have returned to the issue, and discovered that one of the main sources of the PMI is that the antecedent of implicative statements sometimes does not entail the consequent. If the antecedent does entail the consequent, then most of the paradoxes can be resolved. So, the idea of entailment is important here. In the fourth section of this chapter we will discuss the idea of entailment as well as its relation to material implication. And in the fifth section of this chapter a general overview of this research paper will be given.

1.1 Different Types of Implications: There are various types of implications. Let us take a look at the compound statement 'If ABC is a triangle, then it has three sides'. The consequent of this implication follows from the antecedent by the definition of 'triangle'. We call it *definitional implication*. Another statement 'If all humans are mortal and Socrates is human, then Socrates is mortal' can be examined now. Its consequent follows from the antecedent not by the definition of human, but by the logical inference from the antecedent. We call it logical implication. Implications which denote causal connection are labeled as *causal implication*. Thus the implicative statement 'If there is smoke, then there is fire', asserting causal connection between smoke and fire, is a causal implication. The fourth kind of implication is neither definitional, nor is it logical or causal. This is a type of implication which reports a decision of the speaker to behave in the specified way in the specified circumstances. This kind of implications is termed as *decisional implication*. Thus the implicative statement 'If you get good grades in your exams, I will give you a nice shirt as a gift' is an example of decisional implicative statement by which the speaker reports his decision to behave in a specified way if the antecedent condition is fulfilled.[2] There is a fifth kind of implication. This type of implication does not claim any real connection between antecedent and consequent. We use this sort of implication in our everyday conversation as a method of denying the truth of the antecedent by uttering an obviously false consequent. In Copi's words "This sort of conditional is ordinarily intended as an emphatic or humorous method of denying the truth of its antecedent, for it typically contains a notoriously or ridiculously false statement as its consequent."[3] Thus the statement 'If Jones is a logician, then I am a monkey' is an example of this sort of implication. The speaker of this statement wants to state that Jones is not a logician, since he (the speaker) is obviously not a monkey. That is, the speaker of this statement asserts that since the consequent of this statement is false, the antecedent cannot be true. we can name this kind of implication *reductio implication*.

Definitional, logical, causal or decisional implications pronounce that the antecedent is sufficient to produce the consequent. More simply, the antecedent is sufficient for the consequent. We can express this assertion in terms of truth value in this way: all kinds of implications assert that if its antecedent is true, then its consequent must be true also. Thus if the antecedent is true and the consequent is true, then the implication is true. And if the antecedent is true but the consequent is false, then the implication is false. For example, if the antecedent 'If it rains' of the implication 'If it rains, then there will be a flood' is true, and the consequent of this implication 'There will be a flood' is also true, then the whole implication is true. But if the antecedent 'If it rains' is true where the consequent 'There will be a flood' is false, then the whole implication is false. Now if we symbolize the antecedent as *p* and the consequent as *q*, then the implicative statement would be symbolized as $p \supset q$, which means 'If *p*, then *q*'. So, at this moment we are in a position to construct the following truth table:

p	q	$p \supset q$
T	T	T
T	F	F

Table 1

But that is not all. From the fourth century B.C., we have known that truth and falsehood may be distributed in four ways between two propositions. Philo of Megara introduced this idea.[4] According to this idea, the truth value between two propositions may be distributed in the following four ways:

Either both of them are true

or, the first proposition is true and the second proposition is false,

or, the first proposition is false and the second proposition is true,

or, both of the propositions are false.

So the truth table for the implication, $p \supset q$, must be a four-row truth table. But Table-1 is a two-row truth table. It includes only the first two ways of Philonian distribution of truth value between two propositions p (antecedent) and q (consequent). Now, let us try to construct a four-row truth table for $p \supset q$.

p	q	$p \supset q$
T	T	T
T	F	F
F	T	?
F	F	?

Table- 2

The ideas of definitional, logical, causal or decisional implications are not sufficient to determine the truth value of $p \supset q$ in row-3 and in row-4, because in these cases the antecedents are false. But we cannot leave these rows blank, since from the *principle of excluded middle* we know that any statement is either true or false. And, of course, we cannot insert both of the truth values T and F in these rows, since it will be a violation of the *principle of contradiction* (sometimes called 'the principle of non-contradiction) which asserts that no statement can be both true and false. Thus to satisfy the principle of excluded middle and the principle of contradiction, we must add truth values for $p \supset q$ in row-3 and row-4 and that truth value will be either true or false, not both. Now the question is how can we determine truth values for $p \supset q$ for the row-3 and row-4? The idea of reductio implication helps us to complete the truth table for implication. We have seen that by uttering a reductio implication speaker asserts that since the consequent of that implication is false, the antecedent cannot be true. We can translate this view with a little bit technical expression as: 'it is not the case that the antecedent is true, but the consequent is false'. If it happens so, that is, if we find for a given implicative statement that its antecedent is true but its consequent is false, then the given implicative statement will be a false one. Otherwise the statement is true. Thus the third and fourth rows of the truth table have the truth value T in their final column. Hence the complete truth table for the implication will be as follow:

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 3

This table helps us to understand implication and its truth functional nature. Logic is the study of inference, and implication is one of the most central concepts of inference. Implication is also an important concept in mathematics and many other formal sciences, and so it is important for logicians to understand implication. Thus it is very important to understand implication. This table also represents the truth functionality of the implication. It is easily noticed that the truth value of an implication depends only on the truth values of the constituent atomic statements, not on their meaning or material facts. Logic is not metaphysics, nor is it a material science. It is a formal science, and as a formal science logic is concerned with general patterns of inference, not with the particular object or particular fact. In other words, logic does not deal with the content of statements. Rather, it deals with truth functional relationship among the constituent parts of its compound statements. This truth functional characteristic of logic allows it to develop various rigorous tools and axioms. That's why it is important to use truth functional logic in understanding implications.

1.2 Material Implication: Material implication is not merely another kind of implication. The various kinds of implication, discussed above, are various types of speech acts indeed. But material implication is not a speech act. Rather, it is different level of idea about implicative statements. According to this idea an implication does not claim any 'real' connection between antecedent and consequent. In other words, no internal connection between antecedent and consequent is required here. The so-called relation between antecedent and consequent of an implication can only be described by the phrase 'as a matter of fact'. Thus whatever type of implication it is, the meaning of the implication is that 'as a matter of fact, it is not the case that the antecedent is true, but the consequent is false'. This meaning of implication is termed as material implication. Bertrand Russell has given this name to it.[5] Thus the meaning of material implication is the whole meaning of reductio implication, and it is the partial common meaning of every types of implications discussed above. We can examine the causal implication 'If it rains, then there will be a flood.' In what circumstances will we agree that this causal implication is false? Well, if there is rain (true antecedent) but there is no flood (false consequent), then we will agree that the causal implication 'If it rains, then there will be a flood' is false. That is, the meaning of material implication, i.e. 'as a matter of fact, it is not the case that the antecedent is true but the consequent is false' is equally applicable to causal implication. It can also be shown that like causal implication, definitional implication, logical implication, and decisional implication also contain this meaning. Thus the truth condition of material implication captures the common truth functional meaning of all types of implications.

Hence logicians, in standard logic, consider the phrase 'If — then' in the sense of material implication, so that it can cover every types of implications. So, the symbol '$\supset$' (horseshoe) means 'materially implies'. In other words $p \supset q$ means '*p* materially implies *q*', no matter what type of implication is symbolized as $p \supset q$. And, in relevance to the truth value, $p \supset q$ means—

> As a matter of fact it is not the case that *p* is true and *q* is false
>
> Or, It is not the case that *p* is true and *q* is false
>
> Or, it is not the case that *p* and not- *q*
>
> Or, it is not the case that $p \,.\sim q$
>
> And, more symbolically, $\sim(p \,.\sim q)$

That is, $p \supset q$ means $\sim(p.\sim q)$. Of course, $p \supset q$ may also be translated as $\sim p \vee q$, which means 'either p is false or q is true'. The following truth table, however, shows the equivalence among $p \supset q$, $\sim(p. \sim q)$ and $\sim p \vee q$

	p	*q*	$p \supset q$	$\sim(p \,.\sim q)$	$\sim p \vee q$
1	T	T	**T**	T	T
2	T	F	**F**	F	F
3	F	T	**T**	T	T
4	F	F	**T**	T	T

Table 4

<u>1.3 Paradoxes of Material Implication</u>: The truth table (table-3 or table-4) for material implication is attention grabbing. Here row-1 and row-3 assert that 'a true statement is *implied by* any statement whatever', and row-3 and row-4 assert that 'a false statement *implies* any statement whatever'. Thus the statement 'If p is false, then (if p then q) is true' [in symbolic formulation: $\sim p \supset (p \supset q)$] becomes a logical truth, incapable of being false according to the truth functional interpretation. Again, 'If q is true, then (if p, then q) is true' [in symbolic formulation: $q \supset (p \supset q)$] is logically necessary according to the same interpretation. The following truth table indicates the logical truth of these two statements:

	p	q	$\sim p$	$p \supset q$	$\sim p \supset (p \supset q)$	$q \supset (p \supset q)$	$(\sim p \supset (p \supset q)) \equiv (q \supset (p \supset q))$
1	T	T	F	T	T	T	T
2	T	F	F	F	T	T	T
3	F	T	T	T	T	T	T
4	F	F	T	T	T	T	T

Table 5

Although the above truth table displays the logical necessity of the statements $\sim p \supset (p \supset q)$ and $q \supset (p \supset q)$ from the truth functional interpretation of standard logic, many logicians treat these statements paradoxical. These are paradoxical, because either these statements imply conflicting statements or they are implied by conflicting statements. Logicians label these paradoxes as *paradoxes of material implication*. Some non-symbolic concrete example may help us to understand these paradoxes. Let us take the statements:

(i) If 2+2=5, then the earth is round

(ii) If 2+2=5, then the earth is not round

Both of these statements are true since the antecedent '2+2=5' is false. Suppose that 'the earth is round' is true. So, the statement (i) is true (indicated by row-3; F T $\rightarrow$ T), and the statement (ii) is true either (indicted by row-4; F F $\rightarrow$ T). And if the statement 'the earth is round' is false, the statement (i) will be true (indicated by row-4; F F $\rightarrow$ T) and also the statement (ii) will be true (indicated by row-3; F T $\rightarrow$ T). That is, a false statement (in this example: 2+2=5) implies any statement whatever in a true implication. In other words, the false statement '2+2=5' implies both of the conflicting statements 'the earth is round' and 'the earth is not round' at the same time and place. This is some what paradoxical to many logicians.

Again, similarly, the statements—

(iii) If Bangladesh is in Asia, then a triangle has three sides

(iv) If Bangladesh is not in Asia, then a triangle has three sides

are both true since the consequent 'a triangle has three sides' is true, no matter whether 'Bangladesh is in Asia' is true or false. Suppose, 'Bangladesh is in Asia' is false, still the implication is true which is indicated by row-3 (F T $\rightarrow$ T). And, if 'Bangladesh is in Asia' is true, then also the implication is true which is indicated by row-1 (T T $\rightarrow$ T). That is, a true statement (in this example: a triangle has three sides) is implied by any statement whatever in a true implication. In other words, the true statement 'a triangle has three sides' is implied by both of the conflicting statements ‘Bangladesh is in Asia’ and ‘Bangladesh is not in Asia’ at the same time and place. Again, this is somewhat paradoxical to many logicians.

There is another type of paradox concerning material implication. Sometimes, in material implication a contradictory antecedent implies a consequent. In other words, material implication involves contradiction in it although standard logic is committed to preserve the law of non-contradiction. Again I would like to analyze this paradox with a concrete example. Consider the statement, ‘if America attacks Iran and America does not attack Iran, then the price of oil will raise’. The truth functional interpretation of material implication confirms the truth of this statement although it involves the contradiction—‘America attacks Iran and America does not attack Iran’. In symbolic formulation the concerning statement form will be like this, $(p.\sim p)\supset q$. The following truth table proves the necessity of this implication—

p	q	$\sim p$	$(p.\sim p)\supset q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	T

Table 6

We know any statement of the form $(p.\sim p)$ breaks the law of contradiction and hence it is simply inconsistent. It is not sensible that an inconsistent statement implies something consistent. On the contrary, many logicians think that ‘a contradiction 'spreads' to every proposition, and simple inconsistency is equivalent to absolute inconsistency.'[6] Thus the implication of the form $(p.\sim p)\supset q$ is another kind of paradox of material implication. Logicians call this paradox *ex falso quodlebet*.

So, up to this discussion, we have found three paradoxes in material implications. These are: $\sim p \supset (p \supset q)$, $q \supset (p \supset q)$, and $(p.\sim p)\supset q$. But these are not all. There are lots of paradoxes in material implications. The more we examine, the longer will be the list of paradoxes. Among them the following three PMI are well-known—

1. $p\supset(q\supset p)$ [positive paradox]
2. $p\supset(\sim p\supset q)$ [negative paradox]
3. $(p.\sim p)\supset q$ [*ex falso quodlebet*]

One thing should be noted here that while we discuss about the PMI, we use the term 'paradox' in a wider sense. The term "paradox" (in logic) is defined in *Oxford English dictionary* as: "A statement or proposition which, from an acceptable premise and despite sound reasoning, leads to a conclusion that is against sense, logically unacceptable, or self-contradictory; freq. distinguished by name, esp. of its profounder, or of the type of problem it raises." Now, we see, the PMI are only paradoxes in the wide construal of being 'against sense', they are not 'logically unacceptable' or 'self-contradictory'.[7] The main problem they bear is that they do not match with our commonsense intuitions. We, of course, do not argue that every logically true statement must go with our commonsense intuitions. But it will be more adequate logic of implication if they go with commonsense intuition. From this point of view we are looking for a system of implication which is free from the PMI.

1.4 Entailment: The above mentioned paradoxes of material implications are not new discoveries to logicians. They have been discussing and trying to solve these paradoxes along with some other problems of standard logic for centuries. In the modern period G.E Moore, a friend and colleague of Bertrand Russell, also involved himself in the problems of material implication. He introduced the idea of entailment and uses the term 'entail' instead of 'imply'. The term entail is used to indicate implication where the antecedent and consequent are relevant to each other. Professor Moore says "We shall then be able to say truly that "*p* entails *q*", when and only when we are able to say truly that "*q* follows from *p*" or "is deducible from", in the sense in which the conclusion of a syllogism in Barbara follows from the two premisses, taken as one conjunctive proposition; or in which the proposition 'This is coloured' follows from 'this is red.' '*p* entails q' will be related to '*q* follows from *p*' in the same way in which 'A is greater than B' is related to 'B is less than A.' "[8] Moore uses the symbol *ent.* for entailment. Thus '*p* entails *q*' is symbolized as '*p ent. q*'. In case of entailment, we see, an internal relation, i.e. relevance, between antecedent and consequent is required. Because of this relevance 'this is red' entails the consequent 'this is colored'. One thing cannot be red without being colored. Or, one thing is obviously colored if it is red. There is an internal relation between antecedent and consequent in this sort of implication. Again, because of the lack of relevance 'this is red' does not entail 'Socrates is mortal'. There is no relevance between something's redness and Socrates' mortality. Moore expresses this sort of propositions with regard to their properties. Thus any proposition asserting that a given thing that it has the property P entails the proposition that thing in question also has the property Q, can be re expressed as *x*P entails *x*Q, which means in Moore's words " '*x*P entails *x*Q' is to be true, if and only if the proposition 'AP entails AQ' is true, and if also all propositions which resemble this, in the way in which 'BP entail BQ' resembles it, are true also; where 'AP means the same as 'A has P', 'AQ' the same as 'A has Q' etc. etc."[9] Now, according to Moore, logicians falsely infer that since it is natural to express that *x*P $*$ *x*Q (Moore uses the symbol '$*$' for Russellian symbol '$\supset$') by 'If anything has P, then that thing has Q', it is natural to express AP $*$ AQ by 'If AP, then AQ' and consequently, 'AP implies AQ'. Moore considers that if it is the reason to express '$p * q$' by '*p* implies *q*', then it is obviously a fallacious reasoning.[10] But he fears that a good number of logicians have been considering it true, since it is said by Bertrand Russell. Moore comments that it is an 'enormous howler' of logicians. He says "But I imagine that Mr. Russell himself would now be willing to admit that, so far from being true, the statement that '*q* can be deduced from *p*' means the same as '$p * q$' is simply an enormous 'howler' ".[11]

Whether it is an 'enormous howler' or not, it is clear, from the above discussion that entails and (materially) implies are different relations. The relation (entails) that holds between 'This is red' and "This is coloured' is quite different from the relation (implication) that holds between 'It is Monday' and 'Socrates is philosopher'. To say 'This is red' is to admit that 'It is coloured'. In other words, 'it could not be true that this is red and yet false that this is coloured'. That is, '*p* entails *q*' means 'It could not be the case that *p* is true and *q* is false'. On the other hand, to say that 'It is Monday' is not to admit that 'Socrates is a philosopher'. It can be happen that today is Monday, but Socrates is not a philosopher. The relation between 'Today is Monday' and 'Socrates is a philosopher' is a 'as a matter of fact' relation. That is, '*p* (materially) implies *q*' means 'It is not as a matter of fact the case that *p* is true and *q* is false'. The difference between 'could not be' and 'is not as a matter of fact' is obviously a significant difference between these two types of relations. Lord Susan Stebbing explains it in this way: "[I]f '*p ent. q*' means '*p* could not be true and *q* false', then there is between *p* and *q* a relation such that *q* *follows logically* or *formally* from *p*.

No matter what p and q may be, if 'p *ent.* q', then q can be formally deduced from p. If '$p * q$' means 'p is not as a matter of fact true and q false', then there is not such a relation between p and q that q can be formally deduced from p." [12]

Thus, we see, the idea of entailment is different from the idea of material implication. In the case of entailment the antecedent and consequent should have a relation between them which makes the one a consequent of the other. But in the case of material implication, no such relation between antecedent and consequent is required, i.e. only truth values matter here. All entailments truth functionally involve material implications, but not all material implications involve entailment. Entailment restricts the scope of material implication where there is no 'real connection' or relevance between antecedent and consequent. This lack of relevance is one of the main sources of PMI. In other words, our classical logic of material implication falls short of the important concept of entailment, and because of this shortcoming it contains several paradoxes. Thus, to avoid the PMI, we need to develop a logic of implication grounded in the idea of entailment.

1.5 Overview of the Thesis: Our aim in this research is to find a system of logic which is free from the PMI, and which is grounded in the idea of entailment. Thus an adequate logic of implication must fulfill two criteria—(1) no PMI will be a formula in this system, and (2) the antecedent of an implication, in this system, must entail the consequent. In other words, there must be relevance between antecedent and consequent in any true implication. There are various schools of logic which have been developed to deal with the problem of PMI. In this research paper we would like to focus on three of these main schools of logic. These are many-valued logic, modal logic and relevance logic. We will discuss these schools of logic in chapters 2, 3 and 4 respectively, and try to find out whether they fulfill our criteria. In the fifth and final chapter of this paper we will look back to the whole discussion and try to find out which one of these schools, if there is any, is free from PMI fulfilling the aforementioned criteria.

Chapter 2
Many Valued Logic

Many-valued logics are logical calculi which reject the classical bivalence that there are two possible truth values—true and false—for any proposition. Many-valued logics explore the possibility that some propositions may be neither true nor false. Thus it allows more than two truth values for any proposition. But the law of excluded middle asserts that anything must be either A or not-A. In other words, any proposition must be either true or false, and there is no third option for a proposition but being either true or false. Thus most many-valued logics emerge by rejecting the law of excluded middle, although there are very few exceptions.[13] It should be noted that 'many-valued logics' is not a unique system of logic. Rather, it is a set of various systems of logic. Each element of this set is a logical system that contains more than two values for any proposition. The nature and amount of values varies from author to author according to their motivation. So, there are three-valued logic, four-valued logic, five-valued logic and so on. Of course, none of these schools of many-valued logic emerged as direct response to the PMI. But some schools of many valued logic show some promise in resolving the PMI as they introduce different types of truth tables by rejecting traditional bivalence. In this chapter we will discuss many-valued logic and its usefulness in resolving the PMI along with the historical background of many-valued logic.

2.1 Historical Background: Many-valued logic was introduced as a response to Aristotelian idea about future contingent statements found in his treatise *De Interpretatione*. Aristotelian example of future contingent statement is: 'There will be a sea-battle tomorrow.' when it has not yet been determined whether there will or not will be really a sea-battle tomorrow. The statement is, then, not yet actually true or actually false but potentially either. Epicureans took this idea and rejected the traditional of bivalence which states that every proposition must be either true or false. Medieval logicians took the matter again and some of them develop the idea of *neuter*. They maintain that future contingent statements are neuters, that is, these statements are neither true nor false. Then the three-valued logic (many-valued logic) emerged which asserted three values—true, false and the neuter. The main advocate of this many-valued logic was Peter de Rivo.[14] Emil Post (1897-1954) is one of the first to study many-valued logic in the contemporary period (early 20th century). The other is Jan Łukasiewicz (1878-1956). Post's many-valued logic is entirely formal. On the other hand, Łukasiewicz introduces his many-valued logic for philosophical reasons to provide a more appropriate representation for the indeterminacy of the future.[15] Other logicians, like Moh shaw-Kwei, Kleene, Bochvar contributed a lot to this area of logic. They developed several many-valued matrixes in order to establish the independent axioms in a formal calculus. The number of values in those matrixes varies from three to various infinite sets. It is not possible to discuss all of those many-valued logics in this paper. To serve our purpose, we will discuss only Łukasiewicz's many-valued logic with special emphasis on his three-valued logic and a five-valued logic described by Charles G. Morgan as representative works in this area.

2.2 Łukasiewicz's Many-valued Logic: In 1920 Łukasiewicz published two books under the titles *O Pojeciu Mozliwosci* (*On the Concept of Possibility*) and *O Logice Trojwartosciowej* (*On Three-valued Logic*) in which he originated his many-valued logic. He introduces a third truth value—possibility—which can be ascribed to statements about future events. Łukasiewicz rejects the law of excluded middle, as he establishes a logic containing more than two truth values instead of conventional bivalence. He calls his logic 'many-valued logic' instead of 'non-Aristotelian logic', because it is not clear that Aristotle himself considered the law of excluded middle to be universally valid. We find statements in Aristotle's writings which suggest that the law of excluded middle is not applicable to the statements about the future. On the other hand, Chrysippus, a stoic logician, strongly argued in favor of the universality of the law of excluded middle. That's why Łukasiewicz called many-valued logic 'non-Chrysippean logic' rather than non-Aristotelian.[16] Łukasiewicz introduces a special notation, often called Polish notation, for his logic. It is a parenthesis and punctuation free notation system where the functors are always written before *wff*. Thus *Np* corresponds to ~p, *Cpq* to $p \supset q$, *Apq* to $p \vee q$, *Kpq* to $p \bullet q$ and *Epq* to $p \equiv q$. Any functor or operator (*N. A. C. K. E*) standing for binary logical operators has as its scope the first two *wff* immediately following it. Thus the *wff* $p\supset(q\supset p)$ and $p\supset(\sim p\supset q)$ can be re-expressed in Polish notation as *CpCqp* and *CpCNpq* respectively.

Another thing should be mentioned here once again that the number of values in many-valued logics ranges from three to various infinite sets. Thus Łukasiewicz uses the expression L_n in his many-valued logic where *n* stands for the number of values which the variables of calculus range over. So, L_2 indicates the standard propositional calculus containing the bivalence, true and false, whereas L_3 indicates three-valued logic, L_4 indicates four-valued logic and so on. For fixed finite *n*, the *n* values of L_n are taken for convenience as *n* members of the set $V_n = \{\frac{i}{n-1}\}$ where $0 \le i \le n-1$, $i = 0$ and $i = n-1$ establish that 0 and 1 are elements of every set V_n. Now, if there is more than two values in a many-valued logic, the set of values will consist of a set of *n* rational functions with 0 and 1 as end points. For example, in case of L_3, L_4, L_5 the sets of values will be $V_3 = \{0, \frac{1}{2}, 1\}$, $V_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $V_5 = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ respectively where $\frac{3}{3}$ and $\frac{4}{4}$ are replaced by 1. Defining V_n as a set of *n* rational functions is entirely a conventional way of defining it which facilitates the use of algebraic techniques in proving meta- theorems about the L_n.[17] So, there are different matrixes for different numbers of values. In the next section, we will discuss Łukasiewicz's matrix of L_3 as a representative model of three-valued logic.

Łukasiewicz's L_3 Matrix: Łukasiewicz introduces the third value 'intermediate', or 'neutral', or 'indeterminate' to face the problem of future contingent statements. Future contingent statements which are, in his opinion, neither true nor false, must have a third value other then true or false. In Łukasiewicz's words, "Therefore, the proposition considered is at the moment *neither true nor false* and must posses a third value, different from '0' or falsity and '1' or truth. This value we can designate by $\frac{1}{2}$. It represents 'the possible', and joins 'the true' and 'the false' as a third value. The three-valued system of propositional logic owes its origin to this line of thought."[18] The following truth tables represent Łukasiewicz's line of thought regarding his three-valued logic.

N	1
1	0
0	1
½	½

Table 1

A	1	½	0
1	1	1	1
½	1	½	½
0	1	½	0

Table 2

K	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

Table 3

C	1	½	0
1	1	½	0
½	1	1	½
0	1	1	1

Table 4

Here table1 is for negation, table 2, table 3 and table 4 are for disjunction, conjunction and implication respectively. Now, we see, the conjunction takes the minimum value of the conjuncts while disjunction takes the maximum value of the disjuncts. In case of implication, the table describes that the conditional is false only when the antecedent is true and the consequent is false. And an implication is indeterminate when antecedent is true and consequent is indeterminate, or when the antecedent is indeterminate and the consequent is false. The rational for these choices are that (i) when the antecedent is true and the consequent is indeterminate, the implication could be true if the consequent were true, and it could be false if the consequent were false, (ii) when the antecedent is indeterminate and the consequent is false, the implication could be true if the antecedent were false, and it could be false if the antecedent were true. The table for C, thus, makes 'If *p* then *q*' true as long as *q* is no further from truth than *p* is which preserves the natural idea that a true implication will not lead us away from such truth as we already have.[19] These truth tables reject the law of excluded middle and the law of non-contradiction as well. The calculus, by which the law of excluded middle is rejected, is:

Law of excluded middle: $p \vee {\sim} p$

In Polish notation: A*p*N*p*

$= A \frac{1}{2} N \frac{1}{2}$

$= A \frac{1}{2} \frac{1}{2}$

$= \frac{1}{2}$ (indeterminate)

And the calculus, by which the law of non-contradiction is rejected, is:

Law of non-contradiction: $p . {\sim} p$

In Polish notation: NK*p*N*p*

$= NK \frac{1}{2} N \frac{1}{2}$

$= NK \frac{1}{2} \frac{1}{2}$

$= \frac{1}{2}$ (indeterminate).

Of course, these tables do not falsify the law of excluded middle or the law of non-contradiction. These two laws are turned into indeterminate where as these are true for their every substitution instances in standard two-valued logic. In fact, no law of standard logic are ever actually false in Łukasiewicz's many-valued logic. Some of the laws of standard logic, of course, are sometimes turned into neuter or indeterminate in Łukasiewicz's many-valued logic.[20]

L_3 and the Paradoxes of Material Implication: Three-valued logic (or many-valued logic in general) did not emerge as a response to the paradoxes of material implication (PMI). Rather, it was introduced as a response to Aristotelian ideas about future contingent statements. Thus, we do not find any direct attempt to resolve the PMI in the L_3 system. Of course, the rejection of the law of excluded middle and the law of non-contradiction in L_3 system is sometimes considered as an advance in finding the solution to the PMI. This rejection of these two laws are advance, regarding the solution of the PMI, in the sense that due to these two laws we have been obligated to add truth values—true, false—in the rows 3 and 4 of the table 3 in the first chapter (section 1), and these two truth values (row 3 and row 4 in table 3, chapter 1, section 1) cause the paradoxes. But the rejection of the law of excluded middle, the law of non-contradiction and the introduction of the third value 'indeterminate' is not enough to resolve the problem regarding PMI. If we take a closer look at the truth table for implication in L_3 system, we find that it preserves the intact truth table of implication of conventional bivalence used in standard two-valued logic which is, in fact, the cause of above mentioned paradoxes. The following truth tables disclose this feature of L_3 system:

L_2	1	0
1	1	0
0	1	1

Table 5

L_3	1	0	½
1	1	0	½
0	1	1	1
½	1	½	1

Table 6

It is clearly seen in the above tables that the whole content of the table 5 of standard logic is contained in the table 6 of Łukasiewicz's three-valued logic. As a result L_3 system also contains all the PMI. For example, we can consider here the positive paradox (the first of the PMI in our list, Chapter 1, Section 1). The statement $p \supset (q \supset p)$, which represents the positive paradox, can be translated into Polish notation as *CpCqp*. It can have nine ($3^n = 3 \times 3 = 9$, 3 is the number of values and *n* is the number of components) different substitution instances with different combinations of truth values. Now, we can construct the following calculi for each of the substitution instances of *CpCqp*:

CpCqp = C1 C11 = C11 = 1 [when p=1 and q=1]

= C1 C01 = C11 = 1 [when p=1 and q=0]

= C1 C½1 = C11 = 1 [when p=1 and q=½]

= C0C10 = C00 = 1 [when p=0 and q=1]

$= C0\ C\tfrac{1}{2}0 = C0\tfrac{1}{2} = 1$ [when $p=0$ and $q=\tfrac{1}{2}$]

$= C0\ C00 = C01 = 1$ [when $p=0$ and $q=0$]

$= C\tfrac{1}{2}\ c1\tfrac{1}{2} = C\tfrac{1}{2}\tfrac{1}{2} = 1$ [when $p=\tfrac{1}{2}$ and $q=1$]

$= C\tfrac{1}{2}\ C\tfrac{1}{2}\tfrac{1}{2} = C\tfrac{1}{2}1 = 1$ [when $p=\tfrac{1}{2}$ and $q=\tfrac{1}{2}$]

$= C\tfrac{1}{2}\ C0\tfrac{1}{2} = C\tfrac{1}{2}1 = 1$ [when $p=\tfrac{1}{2}$ and $q=0$]

Thus, we see, for every substitution instance, the truth value of *CpCqp* is 1, it is necessarily true. In other words, like standard logic the positive paradox, *CpCqp,* is a theorem in L_3 system also. It can also be shown by constructing similar calculi that other PMI are also theorems in L_3 system. Hence we do not find any solution of the PMI in Łukasiewicz's three-valued logic.

Although three-valued logic does not solve the PMI, its role in this regard is not negligible. Three-valued logic is significant from a historical perspective. By rejecting the law of excluded middle, the law of non-contradiction and the conventional bivalence as well, three-valued originates other many-valued logics, such as four-valued logic, five-valued logics and so on, in which we can observe some at least indirect attempts of resolving the PMI. Moreover, three-valued logic opens the door to modal logic by establishing the theory that plain true and false are not the only truth values for a given statement. And it must be noted here that modal logic has some direct involvement in resolving the PMI. We will, however, discuss a five-valued system of logic in the next section and will evalute how much help, if any, we achieve from it in resolving PMI.

2.3 Morgan's Five-valued Logic: There are various many-valued logics with different numbers of values. To serve our purpose, here, we will consider a model of five-valued logic described by Charles G. Morgan. In this five-valued logic, we see five different values. These are—T for 'logically true', t for 'empirically true, U for 'undeterminable', f for 'empirically false' and F for 'logically false'. Among those only T and t are designated values and all others are undesignated values. The following truth tables for various connectives are offered in this system—

$\supset$	T	t	U	f	F
T	T	t	U	f	F
t	t	t	U	f	f
U	U	U	U	U	U
f	U	U	U	U	U
F	U	U	U	U	U

Table 7

$\rightarrow$	T	t	U	f	F
T	T	t	f	f	F
t	T	t	t	f	f
U	T	t	t	t	t
f	T	t	t	t	t
F	T	T	T	T	T

Table 8

$\vee$	T	t	U	f	F
T	T	T	T	T	T
t	T	t	t	t	t
U	T	t	U	U	U
f	T	t	U	f	f
F	T	t	U	f	F

Table 9

Here, we see that the conditional '$\supset$' reflects the view that if the antecedent is true, then we cannot decide the value of an implication. This is some what new idea in this context. But the conditional '$\rightarrow$' is parallel to the usual material implication. Among various connectives $\neg$, $\rightarrow$, and $\checkmark$ are *pseudo-classical* operators in the sense that if we assign the value 'true' to all designated values and the value 'false' to all undesignated values, then the truth conditions for these operators correspond to that of classical logic. But the connectives ~ and $\supset$ are not pseudo-classical. They offer a new type of truth values for implication. In the next section we will check whether this new type of truth conditions may help us in resolving PMI.

Five-valued Logic and Paradoxes of Material Implication: The truth table for $\supset$ in the above five-valued model is an advance in resolving the paradoxes in the sense that it turns those paradoxical statements into 'undeterminable' instead of theorems. Let us check, for example, the positive paradox $p\supset(q\supset p)$. It can easily be shown that the truth value of this statement is undeterminable when any of the components of it is false or undeterminable. Let p be false and q be true. We can, now, construct the following matrix according to this system—

$p\supset(q\supset p)$

F$\supset$(T$\supset$F)

F$\supset$F

U (undeterminable)

Thus, although it does not falsify the positive paradox, it turns it into undeterminable. Hence positive paradox, i.e. $p\supset(q\supset p)$, is no longer a theorem in this system. In this way, we can show that other PMI are turned into undeterminable in this system of five-valued logic. Since no PMI is true in five-valued logic, they are not theorems in this system. Moreover, this five-valued logic shows that a true statement implies any true or false or undeterminable statement, whereas a true statement is implied only by a true statement, and a false statement does not imply any true or false statement. Thus the principles which cause the PMI—(a) a true statement is implied by any statement whatever, and (b) a false statement implies any statement whatever—are gone. Rather, this five-valued logic establishes some alternative principles—

1. A true statement is implied *only by* a true statement
2. A false statement *does not imply* any true or false statement

None of these alternative principles include the PMI as theorems in this system. Undoubtedly, it is an advance in resolving these paradoxes. But still there is a big problem. The problem is that this five-valued logic resolves PMI at a very high cost. Not only the PMI, but most of the other theorems containing the connective $\supset$ also become undeterminable in this system. We can consider Modus Ponens (MP) here. It is a theorem in standard logic and acceptable as a valid form of inference intuitively and important from mathematical perspective. But in this five-valued system Modus Ponens also becomes undeterminable instead of being true for every substitution instances. We can construct the matrix in this way: If $p\supset q$ and p, then q (MP); it can be re-translated as $[(p\supset q)\bullet p]\supset q$. Now,

$[(T \supset F) \bullet T] \supset F \equiv U \supset F \equiv U$ (undeterminable), when p is true and q is false. Since MP becomes undeterminable, it is no longer a theorem in this system. Similarly, other theorems containing the 'non pseudo-classical' connectives $\sim$, $\supset$ become undeterminable and hence not theorems in five valued system. Thus the whole system of propositional calculus collapses in this five-valued system of logic. Very few logicians, if any, will agree to accept this system at the cost of so many theorems and propositional calculus.

There is another problem. it is well-know that an argument must be either valid or invalid. There is no third option for an argument between being valid or invalid. Now the question is, what will be the status of an argument from the perspective of validity which has an undeterminable value as its premiss or conclusion or as both? Introducing a third option, such that 'undeterminable argument' or something like that, will not be acceptable. Validity or invalidity of an argument is completely a formal matter. An argument is valid if its premisses deduce the conclusion and invalid if its premisses do not deduce its conclusion. It is unreasonable to think that there may be arguments in which premisses neither deduce the conclusion nor 'not deduce' the conclusion. Thus the option for a third category from the perspective of validity is impossible. To escape from this problem one might wish to categorize this sort of argument, which contains undeterminable value either in its premisses or in its conclusion, as valid or invalid. But this will not do. Suppose, the premiss of the argument Ψ is undeterminable and the conclusion is false. In this case the argument could be valid if the premisses were false, and Ψ could be invalid if the premisses were true. So, we cannot categorize it in either of categories—valid or invalid. Similar problems will arise in the case of arguments which have true premisses and undeterminable conclusions, or which have undeterminable premisses and undeterminable conclusions. Thus five-valued logic destroys the way of judging validity of arguments while it resolves the PMI. It is hard to accept this solution at such a high price.

2.4 Comments: So, we have seen in this chapter that three-valued logic contains all the paradoxes of material implication as theorems, although it introduces a third value—indeterminate. It fails to offer any solution for PMI, because it preserves the truth table of implication of standard two-valued logic. The five-valued logic described by C G. Morgan offers solution of the paradoxes by turning those paradoxical statements into undeterminable instead of theorems. Nonetheless this solution is not convincing, since it takes such a high price that most of the logicians will not agree to accept it. Thus, we are finally obligated to take the unfortunate conclusion in this chapter that many valued logic does not offer any convincing solution to the paradoxes of material implication

Chapter 3

Modal Logic

Modal logic is a branch of logic that deals with expressions containing modal features such as 'necessarily', 'possibly', 'contingently', 'can', 'could', 'may', 'must', 'might', 'have to' and so on. Those expressions are often termed as *alethic* modifiers. This term has come from the Greek word *alethea* which means truth. These above mentioned words are said to express alethic modalities. In other words those expressions express various modes of truth. Modal logic, in its narrow sense, is the study of the syntax and semantics of alethic modalities.[21] Modern modal logic emerged as a response to the PMI. American logician Clarence Irving Lewis was the pioneer of modern modal logic. He introduced the idea of strict implication based on modal features. The idea of strict implication rejects the well-known PMI, $p\supset(q\supset p)$ and $\sim p(q\supset p)$. Thus modal logic has some promise in resolving the PMI, and hence modal logic is relevant to our project. There are various models and systems in modal logics, such as *K, D, T, M, B, S1, S2, S3, S4, S5* etc. Of course, our aim in this chapter is not to make an overall study about various kinds and systems of modal logic. Our aim in this chapter is to search whether there is any solution for PMI in modal logic. So, we will discuss here the nature of modal logic (more specifically, propositional modal logic) in general with emphasis on alethic modal logic along with the historical background of modal logic, and above all Clarence Irving Lewis'(1883-1964) idea about strict implication. These are the topics which are very much relevant to our main project, i.e. finding a solution of PMI, if there is any, within the scope of modal logic.

<u>3.1 Historical Background</u>: Modal logic is not, in fact, a new branch of logic. Rather, in the ancient period philosophers showed considerable interest in this area. Aristotle's treatise *De Interpretatione* consists of two chapters about modality. "Aristotle determines in *De Interpretatione*, for example, that 'it may be' and 'It cannot be' are contradictories, as are 'It may not be' and 'It cannot be'. Furthermore, 'from the proposition "It may be" it follows that it is not impossible' and in one sense 'the proposition "It may be" follows from the proposition "It is necessary that it should be". In another sense (which we might gloss as 'It is merely possible that'), 'It may be is logically incompatible with 'It is necessary that it should be'."[22] Clearly those were the pioneering discussions of modal logic. A similar discussion also occupied a substantial part of Aristotle's another classic *Prior Analytics*. In the middle age it was again studied by the Arab and the Christian logicians. Although modal features (such as necessity, possibility, impossibility and so on) are very important topics in philosophy and always play a significant role in philosophical discourse, modal logic found little place in nineteenth and early twentieth century mathematical logic.[23] It was American logician C. I. Lewis who brought modal logic into the light once again in 1932. Lewis reintroduced modal logic while he was criticizing the two basic paradoxes of material implication (PMI)- $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$ - which are accepted as theorems in Whitehead and Russell's treatise *Principia Mathematica*. Lewis maintains that these two statements are false with respect to more natural *strict* sense of implication. Thus he develops an alternative system based on modal features of propositions, namely *strict implication* (we will discuss about it in section 3.4.1). Lewis offers five different axiom systems for his logic of strict implication. All of these systems are based on the modal features of propositions. After Lewis modal logic is enriched with the contribution of Carnap, Kanger, Montague, Hintikka, Von Wright, Saul Aaron Kripke and others. Some alternative modal systems and models have been developed by them. In fact, modal logic is nowadays one of the most actively pursued branches of contemporary mathematical logic.

3.2 Basic Nature of Modal Logic: It is difficult to give a concise definition of modal logic, because there are many different modal systems and models. The best way of understanding modal logic, thus, is to give some general account of modal notions. The basic modal notions are the ideas of 'necessity', 'possibility', 'impossibility' and 'contingency'. By necessity, here, we mean *logical necessity*. And by logical necessity 'we do not mean that, things being as they are, or the world being as it is, it cannot fail to be true; but rather that it could not fail to be true *no matter how* things were, or no matter what the world turned out to be like.'[24] Thus the statement that 'no body can travel faster than light' is not necessary, although there is scientific evidence that it is not possible for a body to travel faster than light. This statement is not necessary because it is only supported by the facts about the physical universe as it is. But it can be claimed, at least theoretically, that the physical universe might have been other than in fact it is. On the other hand, statements, such as ‘all bachelors are unmarried’ or ‘there is no round square’, are true no matter how things are, or what the world turned to be like. In every case, in other words in any possible world (we will discuss about the idea of possible world in the section 3.3.2), these statements are true. Thus these statements are necessarily true. Similarly, by ‘possibility’, we mean *logical possibility*. By ‘impossibility’, we mean *logical impossibility*, and by ‘contingency’, we mean *logical contingency*. It should be noted here that to understand possibility and necessity, modal logicians use ‘possible worlds idiom’ as a powerful analytic tool of interpretation of modal logic. A possible world is a way that the world might have been. According to this interpretation, if a statement is true in all possible worlds, then it is a necessary truth. If a statement happens to be true in our actual world, but is not true in all possible worlds, then it is a contingent truth. And a statement which is true in some possible worlds (not necessarily our own world) is called a possible truth. These four basic modal notions, i.e. necessity, possibility, impossibility and contingency are closely related to each other, and any of them can be defined or explained by any of the others. For example, to say that the statement *p* is necessarily true can be explained as or can alternatively be expressed as ‘it is not possible that *p* is false'. Similarly, to say that the statement p is possible is to say that 'it is not a necessary truth that *p* is false'. There is, however, another important modal notion—entailment. To say that the statement *p* entails the statement *q*, is simply an alternative way of saying that '*q* logically follows from *p*' or that the inference from *p* to *q* is logically valid.[25] Thus we find five basic modal notions which can be expressed briefly in the following way—

> Necessary: If it could not possibly false.

> Possible: If it might be true (whether it is actually true or actually false).

> Impossible: If it is necessarily false.

> Contingent: If it is not necessarily true, that is, possibly true and possibly false.

> Entailment: '*p* entails *q*' means '*q* logically follows from *p*'

Among the above mentioned modal notions necessity and possibility are the basic modal operators. Logicians introduce a square-shaped symbol, $\Box$ (sometimes *L*), for necessity, and a diamond-shaped symbol, $\Diamond$ (sometimes *M*), for possibility. Thus '*p* is necessary' is symbolized as $\Box p$, and '*q* is possible' is symbolized as $\Diamond q$. The other basic modal operator is ,$\prec$, which is used for logical entailment. Thus '*p* logically entails *q*' is symbolized as $p \prec q$. The symbol $\Box$ and $\Diamond$ are classified as monadic operators as these deal with single statement while the symbol $\prec$ is classified as a dyadic operator, because it expresses a relation between two statements.

We have mentioned earlier that there is a close connection between necessity and possibility. According to that interpretation the statement '*p* is necessary' is equivalent to 'it is not possible that not *p*'. And, similarly, '*p* is possible' is equivalent to the statement 'it is not necessary that not *p*'. Thus we find the following valid equivalences—

$$\Box p \equiv \sim \Diamond \sim p$$

$$\Diamond \mathrm{p} \equiv \sim \Box \sim p$$

Any system containing these equivalences does not need to have both of the primitives $\Box$, $\Diamond$. A system can take $\Box$ as primitive and introduce $\Diamond$ by the definition—

$$\Box\, \alpha =_{\text{def}} \sim \Diamond \sim \alpha$$

Similarly, a system can take $\Diamond$ as primitive and introduce $\Box$ by definition—

$$\Diamond\, \alpha =_{\text{def}} \sim \Box \sim \alpha$$

A system which takes $\Box$ as a primitive is called $\Box$-based (*L*-based) system. And a system that takes $\Diamond$ is primitive is called $\Diamond$-based (*M*-based) system.[26] And, in case of entailment, there is controversy about the correct analysis of it. But one thing is not disputed that whenever *p* entails *q*, it is impossible that *p* should be true without *q*'s being true too. Thus the entailment relation, $\prec$, holds between *p* and *q* *when and only when* it is impossible for *p* to be true without *q*'s being true.[27] So, we have a valid equivalence—

$$(p \prec q) \equiv \sim \Diamond (p \cdot \sim q)$$

Hence, we can define entailment, $\alpha \prec \beta$, in the following way—

$$(\alpha \prec \beta) =_{def} \sim \Diamond (\alpha \cdot \sim\beta)$$

Of course, entailment, $\alpha \prec \beta$, can also be defined as—

$$(\alpha \prec \beta) =_{def} \Box (\alpha \supset \beta)$$

Since, $\sim\Diamond(\alpha \cdot \sim\beta)$ can easily be transformed into $\Box(\alpha \supset\beta)$ by the definition (mentioned above) $\Diamond\alpha \equiv \sim\Box \sim \alpha$ and standard propositional calculus equivalences.

It should be noted here that modal logic includes all the *wff*s of standard propositional calculus with their same interpretation adding modal operators to them. Indeed, C.I. Lewis constructed the first axioms system for modal logic by adding modal operators with *wff*s of propositional calculus. He proposed several *nonequivalent* modal systems using these axioms. Most of the modal systems, which are developed after Lewis, are based on Lewis' systems indeed. Considering the significance of Lewis' systems, we would like to list some of his axioms of which the first eleven are used in developing the modal system S1 through S5.[28] In this list the propositional calculus analogue (sentential logic analogue) of the respective modal axioms are also given.

	Modal Axiom	Sentential Logic Analogue	
(1)	$(p \cdot q) \prec (q \cdot p)$	$(p \cdot q) \supset (q \cdot p)$	(Comm)
(2)	$(p \cdot q) \prec p$	$(p \cdot q) \supset p$	(Simp)
(3)	$p \prec (p \cdot q)$	$p \supset (p \cdot p)$	(Taut)
(4)	$[(p \cdot q) \cdot r] \prec [p \cdot (q \cdot r)]$	$[(p \cdot q) \cdot r] \supset [p \cdot (q \cdot r)]$	(Assoc)
(5)	$p \prec \sim\sim p$	$p \supset \sim\sim p$	(DN)
(6)	$[(p \prec q) \cdot (q \prec r)] \prec (p \prec r)$	$[(p \supset q) \cdot (q \supset r)] \supset (p \supset r)$	(HS)
(7)	$[p \cdot (p \prec q)] \prec q$		
(8)	$\Diamond (p \cdot q) \prec \Diamond p$		
(9)	$(p \prec q) \prec (\sim \Diamond q \prec \sim \Diamond p)$		
(10)	$\Box p \prec \Box \Box p$		
(11)	$\Diamond p \prec \sim \Diamond \sim \Diamond p$		
(12)	$p \prec \sim \Diamond \sim \Diamond p$		
(13)	$\Diamond \Diamond p$		

Among those axioms the system S1 contains the first seven axioms. The system S2 contains the first seven axioms and the axiom no. 8; the system S3 contains the first seven axioms and the axiom no. 9; the system S4 contains the first seven axioms and the axiom no. 10; the system S5 contains the first seven axioms and the axiom no. 11. All theorems of S1 are theorems of S2, and all theorems of S2 are theorems of S3 and so on, but not vice versa. Thus, systems get stronger and stronger from S1 to S5 gradually. Among all those systems S4 and S5 are the most significant modal systems. The following two formulas are derivable from S4—

$$\Box A \equiv \Box \Box A$$

$$\Diamond A \equiv \Diamond \Diamond A$$

These formulas help us to substitute any formula with a string of iterated modal operators (e.g. $\Box \Box \Box \Box \Box A$) by a formula in which the relevant string is replaced by a single occurrence of the modal operators in question. And, similarly, the following formulas are derivable from S5—

$$\Diamond A \equiv \Box \Diamond A$$

$$\Box A \equiv \Diamond \Box A$$

These two formulas help us to replace any formula containing a string of two or more modal operators, whether the same or different (e.g. $\Diamond \Box \Box \Diamond \Box \Diamond A$), with a relevant formula containing only one, that is, the last operator of the string. These procedures of replacement of modal formulas help modal logicians to find out easy procedures of natural deduction in modal system.

3.3 Strict Implication: Now we have come to our focus point of this chapter—strict implication. C. I. Lewis introduced the idea of *strict implication* because of his dissatisfaction with Russell and Whitehead's idea of material implication stated in their treatise *Principia Mathematica*. Lewis says, "However, I was troubled from the first by the presence in the logic of *Principia* of the theorems peculiar to material implication..."[29] In *Principia* the material implication $p \supset q$ is considered false if *p* is true and *q* is false, otherwise the implication is true. Thus, if the consequent of an implicative statement is true, then the implication is true, no matter what is the truth value of the antecedent. Again, if the antecedent of an implicative statement is false, then the implication is true, no matter what is the truth value of the consequent. In other words, Russell and Whitehead's logic includes paradoxical formulas as theorems—(1) a true statement is implied by any statement whatever (symbolically: $p \supset (q \supset p)$), and (2) a false statement is implied by any statement whatever (symbolically: $\sim p \supset (p \supset q)$). These are the theorems which are 'peculiar to material implication' in Lewis' view. These paradoxical theorems are called paradoxes of material implication (PMI).There are many other PMI in Russell and Whitehead's system, but these two are the best known among them. We have already discussed about those paradoxes in the first chapter of this paper (section 1.1.3), and that's why we are not describing those paradoxes here once again. However, Lewis suggests that *p strictly implies q* only if it not merely happens not to be the case that *p* is true and *q* is false but could not to be the case that *p* is true and *q* is false. That is, '*p* strictly implies *q*' does not merely means 'Not (*p* and not-*q*). Rather, it means 'Not possibly (*p* and not-*q*). Thus, '*p* strictly implies *q*' includes modal operator 'possible' (◊). Lewis introduces the new symbol, $\prec$, for his strict implication. So, 'p strictly implies q' is symbolized as: $p \prec q$, and the equivalence will be like this: $p \prec q \equiv \sim \Diamond (p \cdot \sim q)$. Hence, Lewis' strict implication can be defined in the following way—

$$\alpha \prec \beta \equiv_{def} \sim \Diamond (\alpha \cdot \sim \beta)$$

This definition of strict implication is adopted in modal system S4 and S5. Apparently, this interpretation of strict implication resolves the PMI. Here, it is not claimed that the falsehood of the antecedent makes the strict implication true. The claim of strict implication is that although the falsehood of p does not suffice to verify the strict implication 'If p, then q', its impossibility does. Because if p cannot be true at all, it is not possible to have a combination of p's truth with q's falsehood. Similarly, here, it is not claimed that the truth of the consequent makes the strict implication true. In this case, the claim of strict implication is that although the truth of q does not suffice to verify the strict implication 'If p, then q', its necessity does, for if q cannot be false at all, it is not possible to have the combination of q's falsehood with p's truth.[30] Hence, the basic paradoxes are gone. Moreover, it includes the necessity of the consequent. Thus we find at least two advantages of strict implication:

- The paradoxes do not hold for $\prec$. i.e. it is not the case that $\sim p \prec (p \prec q)$ or that $p \prec (q \prec p)$.

- Strict implication captures the idea of necessitation. When p strictly implies q, then the truth of p necessitates the truth of q.

3.4 Paradoxes of Strict Implication: It sounds as if the idea of strict implication resolves the PMI. But the problem is that strict implication, unfortunately, has some paradoxes of its own. We have already seen that (1) an impossible statement strictly implies any statement whatever, and (2) a necessary statement is implied by any statement whatever. These are the two basic paradoxes of strict implication which are analogous to the two basic PMI. Thus, any statement of the form $(p \cdot \sim p)$, i.e. an impossible statement, strictly implies any statement q. Similarly, any statement of the form, $(p \vee q)$, i.e. a necessary statement, is implied by any statement q. These two well-known paradoxes of strict implication can be expressed symbolically in the following way:

$$(p \cdot \sim p) \prec q$$

$$q \prec (p \vee \sim p)$$

Proofs for the Paradoxes of Strict Implication: These paradoxes of strict implication can easily be proved by following some quite ordinary, intuitively valid and non-paradoxical rules. These rules are—

1. *Any conjunction implies (even strictly) each of its conjuncts*. This rule is well-known as *simplification*. In Russellian notation it is expressed as $(p \cdot q) \supset p$, and in Polish notation it can be expressed as *KpqCp*

2. *Any statement, p, implies (even strictly)* $p \vee q$*, no matter what q may be*. This rule is known as *addition*. In Russellian notation it is expressed as $p \supset (p \vee q)$, and in the Polish notation it can be expressed as *CpApq*

3. *The statement* $(p \vee q)$ *and* $\sim p$ *together implied (even strictly) q*. This rule is known as *disjunctive syllogism*. In Russellian notation it is expressed as $[(p \vee q) \cdot \sim p] \supset q$, and it can be expressed in Polish notation as *KApqNpCq*.

4. *Whenever p implies (even strictly) q and q implies (even strictly) r, then p implies (even strictly) r*. This is known as the *hypothetical syllogism*. In Russellian notation it is expressed as $[(p \supset q) \cdot (q \supset r)] \supset (p \supset r)$, and in Polish notation it can be expressed as *KCpqCqrCpr*.

Now by using these rules we can derive any arbitrary statement, q, from any impossible statement of the form $(p \cdot \sim p)$. Here we are presenting the proof of it following I. M. Copi's style of natural deduction:[31]

1.	$p \cdot \sim p$	
2.	p	[1, Simplification; rule 1 mentioned above]
3.	$\sim p$	[1, Simplification]
4.	$p \vee q$	[2, Addition; rule 2 mentioned above]
5.	q	[4,3 Disjunctive Syllogism; rule 3 mentioned above]
6.	$(p \cdot \sim p) \prec q$	[1-5, Conditional proof]

Thus, Lewis' strict implication includes the paradoxical statement $(p \cdot \sim p) \prec q$ as a theorem. This is the paradox which is well-known as *ex falso quodlibet*. A similar proof can be constructed for the contention that the necessary the necessary proposition $(p \vee \sim p)$ follows from any propostion at all, say q. That is, the paradoxical statement $q \prec (p \vee \sim p)$ is also included as a theorem in Lewis' system. Moreover, we can show that some other paradoxical statements, such as $\sim \Diamond p \prec (p \prec q)$, $\Box q \prec (p \prec q)$ etc., are also included in Lewis' systems S4 and S5.

Lewis' Self-defence: It is fact that Lewis himself was not comfortable with these paradoxes of strict implication. But he had no way of escaping from them. He thought that if *ex falso quodlibet* $[(p \cdot \sim p) \prec q]$ was false, then the proof for it was defective. But it was clear to him that the proof was constructed following the proper procedure of natural deduction. So, if there was any defect, it had to be in the rules which were followed in constructing the proof. But there is, in fact, no debate about the acceptance of those rules. Those rules are intuitively valid and are accepted by all systems of logic. Thus Lewis took the decision that there was no defect at all. He thought that some paradoxes, such as *ex falso quodlibet*, are unavoidable properties of implication in general. These unavoidable properties are also properties of his strict implication. And that's why those paradoxes become theorem in his strict implication. So, Lewis satisfied himself by saying, "There was no way to avoid the principles stated by these unexpected theorems without giving up so many generally accepted laws as to leave it dubious that we could have any formal logic at all."[32] Some authors and commentators are also sympathetic to Lewis' position. Thus Hughes and Cresswell comment that it will be more harmful for formal logic to abandon any of these generally accepted and intuitively valid rules than to adopt the paradoxes of strict implication as theorems. They says, "This *derivation*[33] shows that the price which has to be paid for denying that ($p \cdot \sim p$) entails q is the abandonment of at least one of A-D[34]. Frankly, this price seems to us exorbitantly high, since all of A-D seem intuitively sound and the principle that ($p \cdot \sim p$) entails q is at worst an innocent one: it could never lead us astray in practice by taking us from a true premiss to a false conclusion, since no proposition of the form ($p \cdot \sim p$) can ever be true."[35]

3.5 Comments: Thus, we see, Lewis' systems and modal logic as well show some primary success in avoiding the PMI, but these systems include paradoxes of strict implication which are almost same as the PMI. So, modal logic and the device, strict implication, are not really successful in resolving the PMI for what we are trying to. Lewis and some other modal logicians try to console themselves by declaring that there is no way to avoid the paradoxes of strict implication without making formal logic impossible. But the situation is not that much drastic as Lewis and some other modal logicians think. There is a still hope to develop formal logic without adopting these paradoxes as theorems. And, there are some logicians who involve themselves in developing such system for formal logic.

Establishment of relevance logic is the most renowned attempt in finding such logic. Since we have not found any convincing solution of the PMI in strict implication and modal logic as well, we will discuss relevance logic in the next chapter in order to search whether it has any solution to the PMI or not.

Chapter 4
Relevance Logic

Relevance logic is a form of non-classical logic in which relevance between the antecedent and consequent, in case of true implication, is required. The term 'relevance logic' is popularly used by North American logicians. British and Australian logicians generally use the term 'relevant logic' for it. In classical logic and non-classical logic so far discussed there are a lot of formulae in which relevance between antecedent and consequent is not required. In contrast, in relevance logic an argument is valid only if there is some relevant connection between the premisses and the conclusion. Similarly, in this system an implication is true only if there is relevant connection between the antecedent and the consequent. We have seen in our earlier discussion that the main cause of the PMI is that there are implications where there is no relevance between antecedent and consequent. As relevance logic accepts implications which have relevance between antecedent and consequent, it is expected that this system of logic can contribute to resolving the PMI. In this chapter we will discuss relevance logic and see how this system tries to resolve the PMI.

4.1 Historical Background: Relevance logic is a comparatively recent branch of non-classical logic. It was born in 1950s. A.R Anderson and N.D Belnap are the pioneers of this logic. They were inspired by the paper *Begründung einer* *strengen Implikation* (A Foundation for a Rigorous Implication) by Wilhelm Ackermann. By the term 'Rigorous Implication' Ackermann expressed the idea that in case of $A \rightarrow B$, a logical connection holds between A and B, that the content of B is part of the content of A. He rejected some of the valid formulas of classical logic, which are in fact paradoxical, on the ground that the truth of A has nothing to do with the question whether there is a logical connection between B and A.[36] Belnap was fascinated with Ackermann's ideas, and was looking for other logicians, if there were any, who were interested in Ackermann's ideas. Very soon he met Anderson who was equally fascinated with Ackermann and they started working together and develop relevance logic.

4.2 Basic Tenets of Relevance Logic: The basic tenet of relevance logic is that it is possible to eliminate the paradoxes of material implication by introducing 'relevance between antecedent and consequent' as a requirement for a true implication. Among those paradoxes $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$ are well-known. Modal logicians tried to resolve these paradoxes by using the device of strict implication. But strict implication has paradoxes of its own. Among the paradoxes of strict implication $(p.\sim p) \prec q$ amd $q \prec (p \vee \sim p)$ are well-known. Relevance logicians claim that the source of the PMI and the paradoxes of strict implication lies in the fact that in each of them the antecedent seems irrelevant to the consequent. As a result the antecedent does not entail the consequent. Moreover, there are other implications, which are valid in classical logic, that are reasonably proved invalid in relevance logic. For example, the implication:

If the moon is made of green cheese, then Bangladesh is in Asia.

In this type of implication, there is also a failure of relevance. Here the consequent has nothing to do with antecedent. Relevance logic rejects this sort of implication, which commits the fallacy of relevance.

4.3 Semantics in Relevance Logic: At the time of its emergence, relevance logic was criticized for not having semantics. But in 1970s Urquhart, Fine, Routley and Meyer, and others developed semantics for relevance logic. In this section we will discuss the semantics for relevance logic following Urquhart. Urquhart introduces the notion of *pieces information*. "A piece of information is a concept which encompasses but is more general than that of a possible world or an evidential situation."[37] He introduces another concept—*satisfaction relation*. Pieces of information satisfy statements. "The satisfaction relation ($\models$) holds between pieces of information and basic statements of a language by virtue of the meaning of those basic statements."[38] Thus if *a* is a piece of information that consists the fact that Al is older then John and the fact that John is older than Bill, then the piece of information satisfies the statement—

$a \models$ Al is older than Bill

Thus those facts which can be satisfied by pieces of information are called *informational link*. Various types of natural laws, scientific truths, conventions are among informational links. Thus 'all bodies attract is other' can work as an informational link. These informational links provide the truth makers for implicative statements.[39]

4.4 Semantics and Implication in Relevance Logic: The relation between implicative statements and the informational link is transitive. Suppose it is a law of nature that a that $A \rightarrow B$ obtains in a and $B \rightarrow C$ is also hold in a, then it seems that $a \models A \rightarrow C$ although there is no direct informational link between A and C. Thus, implication seems to be transitive by virtue of its meaning.[40] However, pieces of information can be combined together. This procedure is called *fusion*. The fusion of two pieces of information a and b is written as:

$$a \circ b$$

And, of course, $a \circ b$ itself a piece of information. When two pieces of information are fused together, an informational link is applied from one piece of information to the other. For example, if a is a piece of information that all bodies attract other bodies, and b is a piece of information that p and q are bodies, then in $a \circ b$ we have the fact that p and q attracts one another. Thus, putting the connection between informational link and implication together, we can derive the following truth condition for implication:[41]

$$a \models A \to B \text{ if and only if } \forall b\,(b \models A \Rightarrow a^\circ b \models B)$$

That is, when a piece of information, which satisfies the antecedent, is fused with another piece of information, then if the fusion satisfies the consequent, the implication is then true.

4.5 Proof Theory for Relevance Logic: Anderson and Belnap introduce a proof theory for relevance logic which is based on Fitch's natural deduction system. The system is simple. Each premiss or hypothesis in a proof is indexed by number. The various steps in proof are indexed by the number of the premisses which are used to derive the steps. An example of this type of proof may help us to understand this technique:

1. $A_{\{1\}}$	hyp.
2. $(A \to B)_{\{2\}}$	hyp.
3. $B_{\{1,2\}}$	1,2 $\to$E
4. $((A \to B) \to B)_{\{1\}}$	2-3 $\to$I
5. $A \to ((A \to B) \to B)$	1-4 $\to$I

The numbers in brackets in this proof indicate the assumptions used to prove the formula. These numbers are called *indices*. "The idea here is that for an assumption to be counted as helping to generate the conclusion, an index denoting the assumption must be appear in the deduction and at some later point be discharged. This ensures that each premise is really used in the deduction.

This natural deduction system gives an intuitive understanding of relevance in proofs."[42] In this system all assumptions stated must be used and indices keep track of which assumptions are used.

Proof Theory and PMI: Relevance logic was developed to avoid the PMI. But it does not prove that all the PMI are false in every circumstance. But relevance logic's advantage is that it does not force the paradoxes to be true. We can consider, for example, the positive paradox, $A \rightarrow (B \rightarrow A)$. Here is an attempt of constructing a proof for this—

1. $A_{\{1\}}$	hyp.
2. $B_{\{2\}}$	hyp.
3. $A_{\{1\}}$	1, reiteration
4. $B \rightarrow A_{\{1\}}$	2,3 $\rightarrow$I
5. $A \rightarrow (B \rightarrow A)_{\phi}$	1-4 $\rightarrow$I

In the fourth step of this so-called proof, there is an illegitimate move. 2 does not belong to {1}. That is why the second hypothesis cannot be discharged here. Thus the proof is not correct. Hence $A \rightarrow (B \rightarrow A)$ is not a formula in this system. The other PMI are also avoided in this way.

4.6 Comments: Thus, we see, relevance logic helps us avoid the PMI. But relevance logic has a short coming that it is not truth functional. Relevance logic is not truth functional in the sense that the truth value of a statement in this system does not depend on the truth values of its components. In other words, the truth value of $p \rightarrow q$ does not depend only on the truth values of p and q. Rather, it depends on the pieces of information and informational links that satisfy the components. These pieces of information and informational links make antecedent and consequent relevant to each other. A concrete example may help us to understand this point. The statement 'If snow is white, then Rome is in Italy' is false, in this system of logic, because its antecedent and consequent are irrelevant to each other. Although both of the components of this implication are true, the pieces of information and informational link of this statement does not make antecedent and consequent relevant. That's why this implication is considered as false in relevance logic, although it is true in other systems of logic. Thus, it is clear that relevance logic deals with the content of statements, not with just the form. Hence the relevance logic is not truth functional. So, if we accept relevance logic, we have to give up traditional truth table and various truth functional devices of standard logic. For this reason many logicians does not admit relevance logic as a satisfactory system of logic.

But for the present purpose, we can still consider relevance logic as an acceptable system of logic of entailment because it not only rejects all the PMI, but also rejects those silly implications (such as: If Socrates is a philosopher, then snow is white) where antecedents and consequents are not relevant to each other. This system satisfies all the criteria we have sat up for an adequate logic of implication. Despite the above criticism, these advantages make relevance logic more satisfactory than the other system of logic discussed in this paper.

Chapter 5
Conclusion

We have discussed the nature of material implication and the paradoxes related to it. In search of the solution of these paradoxes we have discussed three main schools of modern logic: many-valued logic, modal logic and relevance logic. Our aim was to find an adequate system of logic which is free from these paradoxes. We set up two criteria for such a system of logic: (1) the adequate system of logic of implication should not adopt the PMI as theorems. That is, this system should not prove those paradoxical statements as true, and (2) In this system, there must be relevance between the antecedent and consequent of a true implicative statement, so that the antecedent entails the consequent. Now, we will look back to our previous discussion and try to find out which system, if there is any, does satisfy these criteria.

First, we have to look back towards many-valued logic. In our paper, we have discussed two types of many-valued logic—three-valued logic and five-valued logic. Three-valued logic rejects traditional bivalence, the law of excluded middle and the law of non-contradiction by introducing a third value—the indeterminate. Further, the three-valued logic developed a new type of truth tables adopting this third value. The rejections of traditional bivalence, the law of excluded middle and the law of non-contradiction are advances regarding the solution to the PMI in the sense that those conceptions are some of the main causes of the PMI, as we saw in the Chapter 1. However the rejections of traditional bivalence, the law of excluded middle and the law of non-contradiction. and the introduction of indeterminate value are not enough to resolving the PMI. Although three-valued logic rejects traditional bivalence, it adopts the intact truth table of implication based on traditional bivalence in its new truth table. And we know that the truth table of implication in standard logic causes the PMI. By adopting this truth table three-valued logic, in fact, adopts all the PMI in it. That is why all the PMI, such as $p\supset(q\supset p)$, are theorems in this system. So, it fails to resolve the PMI. But we must admit three-valued logic's historical significance in this regard as it is the first attempt, in the modern period, to resolve such logical problems beyond standard logic.

Five-valued logic, another school of many-valued logic, shows more promise than three-valued logic in resolving the PMI. It develops different types of truth tables with five different values. One success of five-valued logic is that all the PMI are valued as 'indeterminate' instead of 'true' in this system, although it does not falsify those paradoxes. So, the PMI are not, at least, theorems in this system. From this point of view, five-valued logic satisfies our first criterion of an adequate logic of implication. But still there is a big problem. Five-valued logic also makes 'indeterminate' many other well-established, mathematically significant and intuitively valid theorems, such as modus ponens, along with PMI. In fact, almost all the theorems containing the connectives ~ and ⊃ become 'indeterminate' and hence not theorems in five-valued logic. Thus the whole procedure of propositional calculus breaks down in this system. Moreover, five-valued logic is not compatible with the validation process of arguments. An argument is invalid if its premisses are true but the conclusion is false. Otherwise the argument is valid. Now, if either the premisses or the conclusion of a given argument is 'indeterminate', there is no way of judging its validity/invalidity. In other words, five-valued logic destroys the way of judging the validity/invalidity of arguments. So, we see, five-valued logic collapses the whole system of propositional calculus and the system of judging the validity of arguments, while it tries to resolve the PMI. It is difficult to accept five-valued logic at such a high price. It should also be noted here that five-valued logic does not offer any device to establish relevance between the antecedent and consequent of implicative statements. That is, it does not make sure that in a true implication the antecedent will entail the consequent. So, it does not satisfy our second criterion for an adequate system of the logic of implication. Hence, we reject five-valued logic as an adequate system of logic of implication.

Modal logic is also inadequate. It emerged as a response to a dissatisfaction with the two basic PMI, $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$, which are accepted as theorems in Whitehead and Russell's treatise *Principia Mathematica*. C.I. Lewis, the pioneer of modern modal logic, maintains that these PMI are false with respect to a more natural and strict sense of implication. He established the idea of strict implication using modal features, which removed the PMI. So, it fulfills our first criterion for an adequate logic of implication. Moreover, the device of strict implication captures the idea of necessitation in the sense that where *p* strictly implies *q*, then the truth of *p* necessitates the truth of *q*. That is there is some sort of relevance between antecedent and consequent. And for this relation of necessitation, it can be said that in strict implication the antecedent entails the consequent. Thus strict implication satisfies our second criterion for adequate logic of implication.

Thus, it sounds that the strict implication or modal logic resolves the PMI. But the unfortunate fact is that, while it resolves PMI, it creates new types of paradoxes, namely the paradoxes of strict implication. Two basic paradoxes of strict implication are $(p.\sim p) \prec q$ and $q \prec (p \vee \sim p)$. In other words, in this system (1) an impossible statement strictly implies any statement whatever, and (2) a necessary statement is implied by any statement whatever. Clearly these two paradoxes of strict implication are analogous to the two basic PMI, $p \supset (q \supset p)$ and $\sim p \supset (p \supset q)$. Thus the strict implication of modal logic does not resolve the PMI in a true sense. Of course, Lewis himself was not comfortable with these paradoxes of strict implication, but he failed to escape from them. Moreover he showed that there were valid procedures of derivation which proved these paradoxes of strict implication as valid while depending only on some generally accepted and intuitively valid rules. Lewis argued that if we do not want to accept these paradoxes of strict implication as theorems, then we have to abandon one or some of these generally accepted and intuitively valid rules. But this abandonment, according to Lewis and many other modal logicians, will destroy the possibility of formal logic. Thus they consoled themselves by declaring that there is no way to avoid the paradoxes of strict implication without making formal logic impossible.

Relevance logicians do not consider the situation as drastic as Lewis and others think it to be. Those logicians hope to develop a system of formal logic which will not adopt the PMI as theorems. They develop relevance logic which is our next school of logic to be considered. The fundamental diagnosis of relevance logic is that the main source of PMI is that in each case the antecedent seems to be irrelevant to the consequent. For this lack of relevance the antecedent does not entail the consequent. Relevance logicians claim that it is possible to to solve the PMI by introducing relevance between antecedent and consequent. To establish this relevance between antecedent and consequent relevance logicians develop various devices, such as pieces of information, informational links and fusion. These devices connect antecedent with consequent by transmitting information. In other words, those devices establish the relevance between antecedent and consequent. Thus, in this system the antecedent entails the consequent in any true implication. So, relevance logic fulfills our second criterion for the adequate logic of implication.

Relevance logic also developed an alternative proof theory based on Fitch's natural deduction system. This system ensures that each premiss is really used in a deduction. In this system all assumptions stated must be used in a deduction. There is a new device, indices, that keeps track of which assumptions are used. This proof theory does not directly disprove the PMI, but it does not allow the PMI to be proved valid. In other words, by this proof theory no PMI can be proved as valid. Thus no PMI is a theorem in relevance logic. In this way the PMI are avoided. So, relevance logic fulfills our first criterion for an adequate logic of implication. Moreover, relevance logic reasonably rejects other kinds of implications which are not paradoxical and are accepted as valid in standard logic, but, in fact, are very silly. The implicative statement, 'If 2+2=4, then Dhaka is the capital of Bangladesh' is that kind of silly implication in which there is no relevance between antecedent and consequent. The uniqueness of relevance logic is that it also rejects this kind of silly implications. It can be said that although those silly implications are not paradoxical, they are fallacious. They commit the fallacy of relevance.

Thus, we see, relevance logic satisfies both of the criteria we have set up for an adequate logic of implication. Not only that, it does something more by rejecting silly implications. Unlike many-valued logic, it does not adopt the PMI or destroy the process of judging the validity/invalidity of arguments. And unlike modal logic and the device of strict implication, it does not create any new type of paradox while it resolves the PMI. Thus relevance logic is a more acceptable logic of implication than any other system of logic we have discussed. We, of course, do not claim that relevance logic is perfect. There are some problems with it. We have already seen that relevance logic is not a truth functional logic. It deals with the content, not with just the form. So, if we accept relevance logic, then we have to give up traditional truth tables and some other devices based on the truth functional characteristic of formal logic. Moreover, although relevance between the antecedent and consequent of an implication is the basic requirement in relevance logic, no criterion has been set up in relevance logic by which one can judge whether any information is really relevant to making the link between the antecedent and consequent. Relevance logicians say that natural law, scientific truth and conventions are among informational links. But these are examples, not really criteria for relevant informational links. We need clear definition and criteria for informational links by which we can measure whether a piece of information or informational link is relevant to connecting the antecedent and consequent. This type of short comings of relevance logic does not, however, make it unacceptable. We have already seen that relevance logic fulfills the criteria for an adequate logic of implication without introducing any new paradox or without creating any new major problem. We should also remember that relevance logic is one of the most recent branches of non-standard logic. More research is going on in this field, and, we hope that relevance logicians will be able to resolve those minor problems very soon. So, it is relevance logic which has the potential to be a perfect logic of implication by resolving all the paradoxes of material implication.

Notes and References:

[1] It should be acknowledged here that many ancient and medieval logicians were concerned about the problem of the paradoxes of material implication and tried to solve it. Aristotle's view of future contingent statements and epicurean's rejection of traditional bivalence opened the door of the development of many-valued logic and modal logic which are concerned to PMI. Stoic logicians, such as Philo, Diodorus and Chrysippus, showed their awareness about the problem of the PMI. Diodorus sat up the criterion—a conditional proposition is if it neither was nor is possible that its antecedent is true and its consequent is false. This criterion has similarity with the modern concept of modal logic. Chrysippus also maintained that some propositions are possible, some imossible, some necessary, some unnecessary. Clearly, this idea matches with the views of modern modal logic. Sometimes it is also claimed that there were echoes of relevance logic in Chrysippus views. Although we admit that ancient and medieval logicians developed some views about many-valued logic, modal logic and relevance logic concerning the problem of the PMI, in this paper we do not discuss their views in details. We have discussed, in this paper, the matter from the perspective of modern logic, since modern logic allows us to continue a more thorough investigation of the problem by using rigorous tools and axiomatic concepts of the propositional calculus developed by Frege and post-Fregian logicians. (See: Mates, Benson(1961), *Stoic Logic*, Berkley and Los Angeles: University of California Press; and Bobzien Susanne, *Dailectical School*, Retreived from http://plato.stanford.edu/entries/dialectical-school, Retreived on July 21, 2005.)

[2] I have classified various types of implication following I.M. Copi, see: Copi, Irving M. and Cohen, Carl (1994). *Introduction to Logic* (ninth Edition), New York: Macmillan Publishing Company, pp.337-40

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[5] In his *Principles of Mathematics* B. Russell called it 'material implication', but in *Principia Mathematica* and in *Introduction to Mathematical Philosophy* he called it simply 'implication' instead of 'material implication'. (see: *The Principles of Mathematics*. London: Routledge, 1903, Reprinted 1992, pp 10-41 & *Introduction to Mathematical Philosophy*. London: George Allen & Unwin Ltd.1919, 14th impression, 1975, pp 144-154)

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[7] I have developed this idea from Professor Jay Foster's (department of Philosophy, MUN) comments. I am indebted to him for this idea.

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[10] Ibid, p.297

[11] Ibid, pp.303-04

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[17] Ackermann, Robert (1967) *An Introduction to Many Valued Logic*, London: Routledge & Kegan Paul Ltd, pp.38-39 (Ackermann used the letter *m* for number of values, but in this paper I have used the letter *n* instead of using *m* for number of values for convenience.)

[18] "Philosophische bemerkungen zu mehrwertigen systemen des Assagenkalkuls", *Comptes rendus des seances de la societe des sciences et des lettres de Varsovie*, Classe III, Vol. xxiii (1930), pp.51-77. An English translation of the paper is given in S. McCall (ed.), *Polish Logic 1920-1939* (Oxford 1967), pp.40-65

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[23] Ibid

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[25] Ibid, p.23

[26] Ibid. p.26

[27] Ibid, p.26

[28] This list of axioms has been taken from: Kahane. Howard (1973), *Logic and Philosophy*, California: Wadsworth Publications Company, Inc., p.348

[29] Lewis, C. I (1930), 'Logic and Pragmatism' in *Contemporary American Philosophy*, G. P. Adams and W. P. Montague (eds.), London: Allen and Unwin, p.32

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[31] Copi, Irving M. (1973), *Symbolic Logic* (Fourth Edition), New York: The Macmillan Company, pp.30-63

[32] Lewis. C. I (1930), op.cit., p.38

[33] My italic, the term 'derivation' refers to the natural deduction showed in section 4

[34] In this context these are the rules 1-4 described in section 4

[35] Hughes. G.E & Cresswell, M.J (1968), op.cit., p.338

[36] *Routledge Encyclopedia of Philosophy* (1998), CD version (version 1.0), London and New York: Routledge, Topic: Relevance logic and entailment

[37] Mares, Edwin, "Relevance Logic" in Jacquette, Dale (Edit. 2002), op.cit. p.612

[38] Ibid, p.612

[39] Ibid, p.612

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[42] Maries. Edwin. "Relevance Logic" in *Stanford Encyclopedia of Philosophy* (On line version), Retrieve date June 15, 2005 from http://plato.stanford.edu/entries/logic

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